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The Restriction of the Ising Model to a Layer

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We discuss the status of recent Gibbsian descriptions of the restriction (projection) of the Ising phases to a layer. We concentrate on the projection of the two-dimensional low-temperature Ising phases, for which we prove a variational principle.

KEY WORDS: Non-Gibbsian states; variational principle; projections; weakly Gibbsian measures.

1. INTRODUCTION

In this paper we study the restriction of the two-dimensional Ising model to a (one-dimensional) layer..The restriction of the plus (or minus) phase is known to be non-Gibbsian below the critical temperature, see refs. 36 and 10. Following suggestions of Dobrushin it was recently shown that this restriction is in fact weakly Gibbsian, see refs. 5, 6, 35, and 31. We state and discuss various recent results on these restrictions. Using elementary methods, we rederive results on weak Gibbsianness below the critical temperature and the results on Gibbsianness above the critical temperature or in a magnetic field. We also add the result that further random decimations of the weakly Gibbsian restriction are again Gibbsian. Finally, we prove the existence of thermodynamic functions (energy and free energy density) and we discuss a variational principle for the weakly Gibbsian measure.

The study of restrictions of Gibbs measures (on $d+1$-dimensional configurations) to a sublayer (of dimension d) can be motivated in various

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ways. First of all they are interesting test cases for an extended Gibbsian description of non-Gibbsian states. Since about ten years various examples of non-Gibbsian states have been produced. Some of these go back to the work of Griffiths, Pearce and Israel,[17, 20] and have become (in)famous as so called renormalization group pathologies.[8] Dobrushin's program tries to understand the non-Gibbsianness as coming from a perhaps too strict requirement on the potential. If, like is the case for unbounded spins, one asks for a potential which is summable on what are typical configurations for the state, one can get at least some effective interaction or physically relevant parametrization of e.g., the images under transformations of states.

A second motivation comes from the theory of interacting particle systems. One of the questions is to see under what conditions the invariant measures of a dynamics are Gibbsian. The simplest scenario is found for so-called probabilistic cellular automata (PCA). These are stochastic dynamics for lattice spin systems under which the spins are updated synchronously in discrete time. If one starts a PCA (with positive transition probabilities) from an invariant measure, then the distribution of space-time configurations (configurations on the space-time lattice) turns out to be a Gibbs measure see ref. 25. Therefore, the invariant measure itself is a restriction of a $d+1$-dimensional Gibbs measure to a d-dimensional hypersurface.

In fact, the question in ref. 36 about the Ising restrictions came quite naturally after it was found in ref. 25 that for high noise dynamics the unique stationary state is Gibbsian as follows from considering the restriction of a Gibbs state in the regime of complete analyticity. The question about the Gibbsian nature of stationary states has of course been considered before, see e.g., refs. 23, 22, 33, 34 and references therein. Understanding the locality of the time reversal operation with respect to the stationary state plays a crucial role in these. While most of these problems are still open (for general PCA's), we feel that the Dobrushin program gives new inspiration towards a (weakly) Gibbsian description of these invariant measures if considered as restrictions of Gibbs measures.

A third motivation can be found in the study of surfaces and models on half-planes with random boundary conditions (wetting phenomena). The restriction of the two dimensional Ising model to a layer can of course be seen as a surface state with respect to the two dimensional Ising measure. The problem of finding an interaction for this restriction consists in finding the interacting between spins at the boundary of an Ising sample (as a function of the configuration at this boundary). The relation with wetting is of a more technical nature. It turns out that in the study of the restriction of Ising measures, one is quickly confronted with questions like how far the influence of a configuration on a sublayer is felt in the bulk of the system. The wetting-context can be used to get a very useful intuitive picture of why, at low temperatures, the restrictions of the plus phase of the Ising model are not Gibbsian. This was made precise in ref. 8, see also ref. 10. An interesting further question (related to the convergence properties of the potential for the restriction) is to see whether there is good decay of correlations close to the surface when on this surface we impose a "typical surface-configuration" (i.e., a sample of the restriction). We will answer this question in Section 4 (Proposition 4.1).

The paper is organized as follows: in Section 2 we introduce basic notations and definitions. We introduce the "telescoping" potential (à la Kozlov,[21]) in Section 3 and discuss its summability properties. In Section 4 we give an overview of the results on the restrictions of the Ising model. In Section 5 we prove the results of Section 4 using the telescoping potential, and finally in Section 6 we discuss the variational principle.

2. DEFINITIONS AND NOTATIONS

2.1. Configuration Space

We consider the regular r-dimensional lattice $\mathbb{Z}^r$ and denote by $\mathscr{L} := \{V, |V| < \infty\}$ the set of finite subsets of $\mathbb{Z}^r$. The complement of a set $V \subset \mathbb{Z}^r$ is $V^c = \mathbb{Z}^r \setminus V$. For two sites $x, y \in \mathbb{Z}^r$ we define

$$|x - y| := \sum_{i=1}^{r} |x_i - y_i| \tag{2.1}$$

The state space is $\Omega := \{+1, -1\}^{\mathbb{Z}^r}$ and its elements (=configurations) are denoted by greek letters $\eta, \omega, \sigma, \xi,\dots$. The value of ω at a site $i \in \mathbb{Z}^r$ is written as $\omega(i)$. For $V \in \mathscr{L}$, $\sigma \in \Omega$ we define

$$\sigma^V(x) = \begin{cases} \sigma(x) & \text{if} \quad x \in V \\ +1 & \text{if} \quad x \notin V \end{cases} \tag{2.2}$$

For $\sigma, \eta \in \Omega$ we define $\sigma_V \eta_{V^c}$ to be the configuration

$$\sigma_V \eta_{V^c}(x) = \begin{cases} \sigma(x) & x \in V \\ \eta(x) & x \notin V \end{cases} \tag{2.3}$$

The restriction of Ω to a volume $V \subset \mathbb{Z}^r$ is denoted by $\Omega_V := \{+1, -1\}^V$ and we write $\sigma_V \in \Omega_V$ for the restriction of a configuration σ.

On Ω we have the natural action of translations τ_a, $a \in \mathbb{Z}^r$ defined by $\tau_a \eta(i) := \eta(i - a)$, $i \in \mathbb{Z}^r$. The σ-algebra generated by the evaluation maps

X_i, $X_i(\omega) := \omega(i)$, $i \in M$ is written as $\mathscr{F}_M = \sigma\{X_i, i \in M\}$. When $M = \mathbb{Z}^r$, we set $\mathscr{F} := \mathscr{F}_{\mathbb{Z}^r}$. The *tail field* σ-algebra $\mathscr{T}^\infty$ is defined as

$$\mathscr{T}^\infty := \bigcap_{V \in \mathscr{L}} \mathscr{F}_{V^c} \tag{2.4}$$

The configuration space Ω is a compact metric space in the product topology. A function f on Ω is called local if it depends only on a finite number of coordinates, i.e., there is a $V \in \mathscr{L}$ such that $f(\eta) = f(\zeta)$ whenever $\eta_V = \zeta_V$. The minimal set V such that this holds is called the *dependence set* of the function.

Definition 2.1. A function $f: \Omega \to \mathbb{R}$ is called *right-continuous* in $\sigma \in \Omega$ if

$$f(\sigma) = \lim_{V \uparrow \mathbb{Z}^r} f(\sigma^V) \tag{2.5}$$

On Ω we have the pointwise order $\eta \preccurlyeq \omega$ if $\eta(x) \leqslant \omega(x)$ $\forall x \in \mathbb{Z}^r$. A function $f: \Omega \to \mathbb{R}$ is called *monotone non-decreasing* if for all $\eta, \xi \in \Omega$, $\eta \preccurlyeq \xi$ implies $f(\eta) \leqslant f(\xi)$.

2.2. Potentials and Specifications

Definition 2.2 (cf. ref. 10). A local specification Γ on $\mathscr{L}$ is a family of probability kernels $\Gamma = \{\gamma_V, V \in \mathscr{L}\}$ on $(\Omega, \mathscr{F})$, such that the following hold:

1. $\gamma_V(\cdot \mid \omega)$ is a probability measure on $(\Omega, \mathscr{F})$ for all $\omega \in \Omega$;
2. $\gamma_V(F \mid \cdot)$ is $\mathscr{F}_{V^c}$-measurable for all $F \in \mathscr{F}$;
3. $\gamma_V(F \mid \omega) = 1_F(\omega)$ if $F \in \mathscr{F}_{V^c}$;
4. $\gamma_{V_2}\gamma_{V_1} = \gamma_{V_2}$ if $V_1 \subset V_2$.

Definition 2.3. A probability measure μ is consistent with a specification Γ (or vice versa), notation $\mu \in \mathscr{G}(\Gamma)$, if $\forall V \in \mathscr{L}$

$$\mu\gamma_V = \mu \tag{2.6}$$

A specification Γ is said to be *translation invariant* if $\forall a \in \mathbb{Z}^r$, $\forall V \in \mathscr{L}$, $\forall \omega \in \Omega$ and for all bounded measurable functions f

$$\gamma_V(f \circ \tau_a \mid \omega) = \gamma_{V+a}(f \mid \tau_a \omega) \tag{2.7}$$

where we abbreviate $\gamma_V(f \mid \omega) = \sum_{\sigma_V} f(\sigma_V \omega_{V^c}) \, \gamma_V(\sigma_V \mid \omega_{V^c})$. We slightly abuse the notation (and circumvent property 3 above) by writing $\gamma_V(\sigma \mid \omega) = \gamma_V(\sigma_V \mid \omega_{V^c}) = \gamma_V(\sigma_V \mid \omega_{V^c})$ if one means to take configurations σ and ω identical on V^c. One should think of $\gamma_V(\sigma \mid \omega)$, as the probability to find σ in V given ω outside of V.

Property 4 of Definition (2.2) is called self-consistency and is most important in characterising equilibrium. It suggests constructing probability measures $\nu \in \mathscr{G}(\Gamma)$ as weak limits of $\gamma_V(\cdot \mid \omega)$, some $\omega \in \Omega$, $V \uparrow \mathbb{Z}^r$ (perhaps along a subsequence). Such weak limits automatically exist by compactness but their consistency with the specification is only immediate if $\gamma_V(f \mid \cdot)$ is a continuous function for all continuous f. One then deals with a so-called Feller (or quasilocal) specification. This is not the context of the Dobrushin program where more general specifications have to be considered and hence that $\mathscr{G}(\Gamma) \neq \varnothing$ is not obvious in general.

In a Gibbsian formalism one considers a special class of specifications, the so called Gibbsian specifications which are of the Boltzmann–Gibbs form:

$$\gamma_V(\sigma \mid \omega) = \frac{1}{Z_V(\omega)} \exp\left\{ - \sum_{A \cap V \neq \varnothing} U(A, \sigma_V \omega_{V^c}) \right\} \tag{2.8}$$

where $Z_V(\cdot) \in \mathscr{F}_{V^c}$ is a normalization factor

$$Z_V(\omega) = \sum_{\sigma_V \in \Omega_V} \exp\left\{ - \sum_{A \cap V \neq \varnothing} U(A, \sigma_V \omega_{V^c}) \right\} \tag{2.9}$$

and $U(A, \cdot)$ is an "interaction potential:"

Definition 2.4. A potential U is a real-valued function on $\mathscr{L} \times \Omega$

$$U \colon \mathscr{L} \times \Omega \to \mathbb{R} \tag{2.10}$$

such that $U(A, \cdot) \in \mathscr{F}_A$ for all $A \in \mathscr{L}$ (put $U(\varnothing, \cdot) = 0$).

A potential U is *translation invariant* if $\forall A \in \mathscr{L}$, $a \in \mathbb{Z}^r S$, $\eta \in \Omega$

$$U(A, \eta) = U(A + a, \tau_a \eta) \tag{2.11}$$

Definition 2.5. 1. A potential U is *convergent* at $\omega \in \Omega$ if for all $V \in \mathscr{L}$

$$\sum_{A \cap V \neq \varnothing} U(A, \omega) \tag{2.12}$$

is well-defined. We always understand an infinite sum $\sum_A a_A$ as well-defined when $\exists a < \infty$ with the property that $\forall \varepsilon > 0 \, \exists V_0 \in \mathscr{L}$ so that $\forall V \in \mathscr{L},\ V \supset V_0$

$$\left| \sum_{A \subset V} a_A - a \right| \leqslant \varepsilon$$

2. A potential U is *absolutely convergent* at $\omega \in \Omega$ if for all $V \in \mathscr{L}$

$$\sum_{A \cap V \neq \varnothing} |U(A, \omega)| < \infty \tag{2.13}$$

3. A potential U is *uniformly absolutely convergent* if for all $V \in \mathscr{L}$

$$\sum_{A \cap V \neq \varnothing} \sup_{\omega \in \Omega} |U(A, \omega)| < \infty \tag{2.14}$$

Let U be a potential and suppose that there exists a set $\bar{\Omega}_U$ in the tail field of points of convergence of U ($\bar{\Omega}_U \in \mathscr{T}^\infty$ and $\forall \omega \in \bar{\Omega}_U$ and $\forall V \in \mathscr{L}$ the sum $\sum_{A \cap V \neq \varnothing} U(A, \omega)$ is well-defined). Then, for every $V \in \mathscr{L}$ and every $\omega \in \bar{\Omega}_U$ we can introduce the finite volume Gibbs measure

$$\mu_V^{\omega, U}(\xi) = \begin{cases} \dfrac{1}{Z_V(\omega)} \exp\left\{ - \sum_{A \cap V \neq \varnothing} U(A, \xi_V \omega_{V^c}) \right\} & \text{if} \quad \xi = \xi_V \omega_{V^c} \\ 0 & \text{otherwise} \end{cases} \tag{2.15}$$

(We ask for $\bar{\Omega}_U$ to be in the tail field to make sure that $Z_V(\omega)$ is well-defined.) Factors of temperature or *a priori* weights (reference measure) are supposed to be contained in the potential. The Dobrushin operator is then defined by taking expectations with respect to (2.15):

$$R_V^U(f)(\omega) := \int f(\xi) \, \mu_V^{\omega, U}(d\xi) \tag{2.16}$$

mapping bounded measurable functions f on Ω to functions $R_V^U(f)$ on $\bar{\Omega}_U$.

Definition 2.6. A probability measure μ on $(\Omega, \mathscr{F})$ is *weakly Gibbsian* if there exists a potential U and a tail field set Ω_U of points of absolute convergence of U (cf. (2.13)) such that

1. $\mu(\Omega_U) = 1$;

2. $\forall V \in \mathscr{L}$ and for every bounded measurable function f,

$$\int_{\Omega} f \, d\mu = \int_{\Omega} R_V^U(f) \, d\mu \tag{2.17}$$

A somewhat less stringent definition of weak Gibbsianness is obtained by asking that there is a tail field set $\bar{\Omega}_U$ of points of convergence of U such that 1 and 2 of Definition 2.6 hold.

If the potential in Definition (2.6) is uniformly absolutely convergent, then v is a (*bona fide*) Gibbs measure.

3. VACUUM AND TELESCOPING POTENTIAL

In this section we shall introduce the so-called telescoping potential which will be a very useful tool in the study of the restrictions of the Ising model. A natural potential associated to a specification is the so-called vacuum potential (in our case the vacuum will always be the configuration of all plusses). To construct this potential, start from

$$H_V(\xi) := \ln \frac{\gamma_V(+ \mid +)}{\gamma_V(\xi^V \mid +)} \tag{3.1}$$

and write

$$H_V(\xi) = \sum_{A \subset V} v(A, \xi) \tag{3.2}$$

This last formula can be inverted (Møbius formula) and we get:

$$v(A, \xi) = \sum_{V \subset A} (-1)^{|A \setminus V|} H_V(\xi) \tag{3.3}$$

The inversion is rapidly checked by remembering that $\sum_{V \subset R \subset B} (-1)^{|R|} = (-1)^{|V|} \delta_{B, V}$ where δ is the Kronecker delta. This potential v is called *vacuum* because it has the property that $v(A, \omega) = 0$ whenever $\omega_i = +1$ for some $i \in A$. This follows easily from (3.3) using that $H_V(\xi) = H_{V \setminus i}(\xi)$ if $i \in V \subset A$ and $\xi(i) = +$. It is straightforward to check that the vacuum potential is the unique potential having this property and that it is translation invariant if Γ is. In ref. 31 it is proved that the vacuum potential v is convergent and consistent with the specification Γ, i.e., $\forall V \in \mathscr{L}$ $\forall \omega$, $\xi \in \Omega$

$$\gamma_V(\xi \mid \omega) = \frac{1}{Z_V(\omega)} \exp\left[- \sum_{A \cap v \neq \varnothing} v(A, \xi_V \omega_{V^c}) \right] \tag{3.4}$$

if and only if the specification is right-continuous.

A possible problem with this vacuum potential is that it may not be *absolutely* convergent (even when the specification is right-continuous). Therefore, in order to obtain absolute convergence Kozlov, in ref. 21, introduces another kind of potential. We now present a simplified version of it which, for the occasion, we would like to call a telescoping potential. For this we turn again to (3.1) and we write

$$\exp(-H_V(\xi)) = \frac{\gamma_V(\xi^V \mid +)}{\gamma_V(+ \mid +)} = \prod_{s=1}^{n} \exp F_s(\xi_{i_1} \cdots \xi_{i_s}) \tag{3.5}$$

where we have lexicographically ordered the $n = |V|$ sites in V according to $i_1 < i_2 < \cdots < i_n$ and

$$F_s(\xi_{i_1} \cdots \xi_{i_s}) := \ln \frac{\gamma_V(\xi_{i_1} \cdots \xi_{i_s} + \cdots + \mid +)}{\gamma_V(\xi_{i_1} \cdots \xi_{i_{s-1}} + \cdots + \mid +)} \tag{3.6}$$

We can now order the sites $i_1,..., i_s = \{ j \leqslant i,\ j \in V\}$ according to their distance from the "largest" site $i_s = i$. For this purpose we consider for every $i \in V$ the sequence of increasing volumes $L_{i,m}$ with

$$L_{i,m} := \{ j \in \mathbb{Z}^r : j \leqslant i,\ |j - i| \leqslant m\}, \qquad m = 0, 1,... \tag{3.7}$$

We thus have the partition

$$\{ j \leqslant i,\ j \in V\} = \bigcup_{m=1}^{m(i,V)} [V_{i,m} \backslash V_{i,m-1}] \cup V_{i,0} \tag{3.8}$$

with $V_{i,m} := L_{i,m} \cap V$ and $m(i,V) \equiv \max_{j \leqslant i,\, j \in V} |i - j|$. Correspondingly, $F_s(\xi_{i_1} \cdots \xi_{i_s}) = F_s(\xi^{V_{i,m(i,V)}})$ can be further telescoped as

$$= -\sum_{m=0}^{m(i,V)} U_{L_{i,m}}(\xi^V) \tag{3.9}$$

with

$$U_{L_{i,m}}(\xi) := \ln \frac{\gamma_{L_{i,m}}(\xi^{L_{i,m-1}} \mid +)\, \gamma_{L_{i,m}}(\xi^{L_{i,m} \backslash i} \mid +)}{\gamma_{L_{i,m}}(\xi^{L_{i,m-1} \backslash i} \mid +)\, \gamma_{L_{i,m}}(\xi^{L_{i,m}} \mid +)} \tag{3.10}$$

for $m > 0$ and

$$U_{L_{i,0}}(\xi) := -F_i(\xi^{L_{i,0}}) = \ln \frac{\gamma_i(+ \mid +)}{\gamma_i(\xi^i \mid +)} \tag{3.11}$$

(Observe that $m(i, V) = 0$ when $i = i_1$ is the "first" site in V). We thus define the (telescoping) potential

$$U(A, \xi) := U_{L_{i,m}}(\xi) \text{ see (3.10), if } A = L_{i,m} \text{ for some } i \in \mathbb{Z}^r,\ m \geqslant 0 \tag{3.12}$$

and $U(A, \xi) \equiv 0$ otherwise.

To get more insight in the potential it is instructive to rewrite it for the one-dimensional case. For $r = 1$ the potential $U(B, \xi)$ is non-vanishing iff $B = L_{i,m} = [i - m, i] \cap \mathbb{Z}$, $i \in \mathbb{Z}$, $m = 0, 1,...$ is an interval. For such a B we rewrite (3.10) as

$$U_{[j,i]}(\xi) = \ln \frac{\gamma_{[j,i]}(\xi^{]j,i]} \mid +)\, \gamma_{[j,i]}(\xi^{[j,i[} \mid +)}{\gamma_{[j,i]}(\xi^{]j,i[} \mid +)\, \gamma_{[j,i]}(\xi^{[j,i]} \mid +)} \tag{3.13}$$

where we abbreviated e.g., $]j, i] \equiv \{j + 1,..., i\}$ for $j < i$ in $\mathbb{Z}$.

Some properties of this potential are immediate. For example, $U_{L_{i,m}} = 0$ whenever $\xi_i = +$ or when $\xi = +$ on the set $L_{i,m} \backslash L_{i,m-1}$. As a consequence and following (3.5)–(3.11), the Hamiltonian (3.1) is telescoped as

$$H_V(\xi) = \sum_{A \cap V \neq \varnothing} U(A, \xi^V) \tag{3.14}$$

Moreover, the potential is explicitly translation-invariant if the specification is.

From now on we will assume that the specification Γ is right-continuous. In the sequel this specification will always be the monotone right-continuous specification (introduced in ref. 10) consistent with the restriction of the Ising model.

In order to verify the consistency of this telescoping potential with the right-continuous specification, we can rewrite U as a resummation of the vacuum potential (see also ref. 21). More precisely

$$U_{L_{i,m}}(\xi) = \sum_{R \ni i,\ R \not\subset L_{i,m-1},\ R \subset L_{i,m}} v(R, \xi) \tag{3.15}$$

for $m > 0$, and for $m = 0$ we have $U_i(\xi) = v(i, \xi)$.

From this, one can check that $\forall V \in \mathscr{L}$

$$\sum_{A \cap V \neq \varnothing} U(A, \xi^V) = \sum_{A \cap V \neq \varnothing} v(A, \xi^V) \tag{3.16}$$

if for every $V \in \mathscr{L}$, U is absolutely convergent in ξ^V (see ref. 21). If there exists a set Ω_U in the tail field such that U is absolutely convergent in every point ω of Ω_U then (3.16) together with the right-continuity of Γ and the consistency of v with Γ (see 3.4) give that

$$\gamma_V(\xi \mid \omega) = \frac{1}{Z_V(\omega)} \exp\left[- \sum_{A \cap V \neq \varnothing} U(A, \xi_V \omega_{V^c}) \right], \qquad \omega \in \Omega_U \tag{3.17}$$

The representation (3.15) of the telescoping potential in terms of the vacuum potential is very useful, because one has a certain freedom in the choice of the sets $L_{i,m}$. The only constraint on these sets is that the obtained potential (from (3.15)) is still consistent with the specification, i.e.,

$$\forall R \in \mathscr{L}\ \exists!(i, m) \text{ such that } R \ni i,\ R \not\subset L_{i,m-1},\ R \subset L_{i,m} \tag{3.18}$$

Indeed, when the constraint (3.18) is satisfied we have

$$\begin{aligned} \sum_{A \cap V \neq \varnothing} U(A, \xi^V) &= \sum_{L_{i,m} \cap V \neq \varnothing} \sum_{R \ni i,\, R \not\subset L_{i,m-1},\, R \subset L_{i,m}} v(R, \xi^V) \\ &= \sum_{A \cap V \neq \varnothing} v(A, \xi^V) \end{aligned} \tag{3.19}$$

and from this, together with the right-continuity of the specification, we conclude that the potential U is consistent with the specification. The constraint (3.18) on the sets $L_{i,m}$ is of a geometric nature. In dimension $r = 1$ we can choose e.g., $L_{i,m} = [i - g(m), i]$, where $g(m)$ is some strictly increasing function. The freedom in the choice of $g(m)$ permits to "tune" a bit the convergence properties of the potential U (for our study of the restrictions it will be okay to choose $g(m) = m$). In $r \geqslant 2$ the sets $L_{i,m}$ introduced before also satisfy this constraint, whereas e.g., $L_{i,m} := \{ j \in \mathbb{Z}^r : |j - i| \leqslant m \}$ do not satisfy the constraint.

In applications, the right-continuous specification Γ is often constructed starting from a probability measure ν such that $\nu \in \mathscr{G}(\Gamma)$ (cf. Definition 2.3). We then know that ν is weakly Gibbsian (cf. Definition 2.6) if there exists a tail field set Ω_U of points of absolute convergence of U such that $\nu(\Omega_U) = 1$. I.e., proving that ν is weakly Gibbsian boils down to showing that

$$\sum_{j \geqslant i} \sum_{m \geqslant |i - j|} |U(L_{j,m}, \xi)| < \infty, \qquad i \in \mathbb{Z}^r \tag{3.20}$$

for a full-measure (tail-)set of ξ's.

The sum (3.20) can only converge if for every $j \in \mathbb{Z}^d$ $U(L_{j,m}, \xi)$ decays fast enough as $m \to \infty$. The typical situation which we meet in the case of projections is that $U(L_{j,m}, \xi)$ decays in fact exponentially in m, for m larger than some function $l(j, \xi)$. The absolute convergence of such a potential is the context of the following proposition.

Proposition 3.1. Let Γ be a local right-continuous specification, $v \in \mathcal{G}(\Gamma)$ and U the telescoping potential defined by (3.10) and (3.11). Suppose that there exist constants C_1, C_2, $M < \infty$, $\lambda > 0$ and a function $l: \mathbb{Z} \times \Omega \to \mathbb{R}^+ \cup \{+\infty\}$ such that $\forall m \geqslant M$, $\forall \xi \in \Omega\ \forall j \in \mathbb{Z}$

$$|U(L_{j,m}, \xi)| \leqslant C_1 I[m \leqslant \ell(j, \xi)] + C_2 I[m > \ell(j, \xi)] \exp[-\lambda m] \tag{3.21}$$

Suppose further that $\exists$ a translation invariant tail field set K, $v(K) = 1$ so that $\forall \xi \in K\ \forall j \in \mathbb{Z}$, $\ell(j, \xi) < \infty$ and

$$|\{j \geqslant i : \ell(j, \xi) \geqslant |j - i|\}| < \infty, \qquad i \in \mathbb{Z} \tag{3.22}$$

Then the telescoping potential U is absolutely convergent for $\xi \in K$ and v is weakly Gibbsian.

Of course, the left hand side of (3.21) is a local function for fixed m while the right hand side can be highly non-local as a function of ξ and deals with the dependence of the potential on ξ as m grows.

Remark 1. Conditions (3.21) and (3.22) may seem weird or ad hoc. In Section 5 we show that they are satisfied for the one-dimensional restriction of the plus-phase of the two-dimensional Ising model. In fact if v is the restriction to a hyperplane of a measure μ then $U_{L_{j,m}}(\xi)$ measures the correlations between spins at sites $(j, 1)$ and $(k, 1)$, $k \in L_{j,m} \backslash L_{j,m-1}$ in a measure $\mu^{\xi L_{j,m}}$ where $\mu^{\xi L_{j,m}}$ is a constrained measure obtained from μ and ξ plays the role of a boundary condition. (This will become more clear in Section 4).

One should think of the $\ell(j, \xi)$ as the radius of a ball around site j outside which the spins at sites $(k, 1)$, $k \leqslant j$ are only weakly correlated to the spin at site $(j, 1)$ in the constrained measure μ_β^ξ of (4.7).

To satisfy (3.22) on a set of full measure for v it suffices that $v[\ell(j, \xi) > u] \leqslant \exp[-cu]$ for some $c > 0$.

Remark 2. For $r = 1$, it is convenient to use the left/right symmetry of the sets $L_{j,m}$ which are just lattice intervals $[i, j]$. We then ask (3.21) (which looks to the left) together with the existence of finite $\ell^+(i, \xi)$ for which

$$|U([i, j], \xi)| \leqslant C_1 I[j - i \leqslant \ell^+(i, \xi)] + C_2 I[j - i > \ell^+(i, \xi)] \times \exp[-\lambda\, |j - i|] \tag{3.23}$$

(looking to the right). The assumption (3.22) in Proposition 3.1 can be replaced by the requirement that for each $i \in \mathbb{Z}$, $\xi \in K$, there are finite $\ell^+(i,\xi)$, $\ell^-(i,\xi) \equiv \ell(i,\xi) < \infty$ so that for all $j > i \in \mathbb{Z}$, (3.21) and (3.23) hold. The idea is that $\ell^-(i,\xi)$ looks in the configuration ξ to the left of i while $\ell^+(i,\xi)$ looks to the right.

Proof of Proposition 3.1. We have to check (3.20). Inserting (3.21) there are two sums to control. The sum for $m > \ell(j,\xi)$ is easily taken care of using the exponential decay. The sum over $m \leqslant \ell(j,\xi)$ has only a contribution if $|j-i| \leqslant \ell(j,\xi)$. We thus get that (3.20) is bounded by

$$C_1 \sum_{j \geqslant i,\, |j-i| \leqslant \ell(j,\xi)} \ell(j,\xi)^r + C_3 \sum_{j \geqslant i} \exp[-\lambda\, |j-i|] \tag{3.24}$$

Using assumption (3.22) this is finite and the conclusion follows from the remarks above. ∎

4. THE RESTRICTED ISING MODEL

4.1. The Model

We consider the standard ferromagnetic nearest neighbor Ising model on the regular $d+1$-dimensional lattice $\mathbb{Z}^{d+1}$, $d \geqslant 1$. The symbols Λ, Λ_n,... will be reserved to indicate finite subsets of $\mathbb{Z}^{d+1}$. Their complement is $\Lambda^c = \mathbb{Z}^{d+1} \setminus \Lambda$ etc. The configuration space for the Ising model is $\Omega' = \{+1, -1\}^{\mathbb{Z}^{d+1}}$. Fix $\beta \geqslant 0$, $h > 0$. For a finite box $\Lambda \subset \mathbb{Z}^{d+1}$ with free (or empty) boundary conditions, the Gibbs state $\mu^h_{\Lambda,\beta}$ for the Ising model assigns a probability

$$\mu^h_{\Lambda,\beta}(\sigma_x, x \in \Lambda) = \frac{1}{Z^h_{\Lambda,\beta}} \exp\left[\beta \sum_{\langle xy \rangle \subset \Lambda} (\sigma_x \sigma_y - 1) + h \sum_{x \in \Lambda} \sigma_x\right] \tag{4.1}$$

to an Ising spin configuration $\sigma_x = \pm 1$, $x \in \Lambda$. The normalization $Z^h_{\Lambda,\beta}$ is the partition function (for free boundary conditions). The parameter β is (proportional to) the inverse temperature and h is called the magnetic field. The first sum in (4.1) is over the nearest neighbor pairs $\langle xy \rangle$ in Λ. Each site $x \in \mathbb{Z}^{d+1}$ has $2(d+1)$ nearest neighbors and we write $y \sim x$ if the site $y \in \mathbb{Z}^{d+1}$ is a nearest neighbor of x. The infinite volume Ising state $\mu^h_\beta = \lim_\Lambda \mu^h_{\Lambda,\beta}$ is obtained in the thermodynamic limit as $\Lambda \uparrow \mathbb{Z}^{d+1}$ along a sequence of sufficiently regular volumes. We can take the (weak) limit $h \downarrow 0$ of $\mu^h_{\Lambda,\beta}$ and we write $\mu^+_\beta = \mu^{0^+}_\beta$ for this limit. In the same way, starting with $h < 0$ and letting $h \uparrow 0$ we can define μ^-_β. We refer to all of these as Ising states. When making no distinction between them, we denote these states by the common symbol μ_β and the corresponding random field is denoted $X=(X(x), x\in\mathbb{Z}^{d+1})$. They are translation invariant probability measures on $(\Omega', \mathscr{F}_{\mathbb{Z}^{d+1}})$ and they all satisfy the Dobrushin–Lanford–Ruelle equation

$$\mu_\beta[X(x)=\sigma_x \mid X(y), y\in\mathbb{Z}^{d+1}\setminus\{x\}](\sigma)=\frac{1}{1+\exp(-2\beta\sigma_x\sum_{y\sim x}\sigma_y-2h\sigma_x)} \tag{4.2}$$

μ_β almost surely. For $h=0$ and for sufficiently large β there are other solutions to (4.2) (even non-translation invariant ones if $d\geqslant 2$) but we will restrict us in what follows to the Ising states introduced above. In particular, there is a critical value $0<\beta_c<\infty$ for which $\mu_\beta^+\neq\mu_\beta^-$ whenever $\beta>\beta_c$. The standard methods, results and more details about the Ising model can be found in almost any textbook on statistical mechanics, see e.g., refs. 38, 12 or 37.

We now fix a hyperplane or layer

$$\Xi=\{x=(x_1,..., x_d, x_{d+1})\in\mathbb{Z}^{d+1}: x_{d+1}=0\} \tag{4.3}$$

which we can identify with $\mathbb{Z}^d$. The sites in Ξ are denoted by $i, j, k,...$, which, though treated as elements of $\mathbb{Z}^d$, ought to be identified with $(i,0), (j,0), (k,0),...$ when considering $\Xi\subset\mathbb{Z}^{d+1}$. Finite subsets of Ξ are written as $V, V_n, A...$ and by V^c we mean the complement of V in Ξ. On Ξ we have a new configuration space $\Omega=\{+1,1\}^{\mathbb{Z}^d}$ with elements $\xi, \zeta, \omega,...$ and we write $\mathscr{F}=\mathscr{F}_\Xi$. Of course, every $\sigma\in\Omega'$ gives rise to a unique $\xi=\xi(\sigma)\in\Omega$ via $\xi_i=\sigma_{(i,0)}$, $i\in\mathbb{Z}^d$ and much of the structure and notation of the spin system on Ξ is inherited quite straighforwardly from that on $\mathbb{Z}^{d+1}$. For example, given $\xi\in\Omega$, we put $\xi^V=\xi(\sigma^\Lambda)=(\xi(\sigma))^V$ for $\Lambda\cap\Xi=V$ and $\sigma_{(i,0)}=\xi_i$. We also write $\zeta_V\xi_{V^c}$ for the configuration which is equal to ζ on V and is equal to ξ on V^c when given $\xi, \zeta\in\Omega$.

This paper is about the restriction of the Ising states μ_β^h and $\mu_\beta^\pm$ to this layer Ξ. In other words, with $(X(x), x\in\mathbb{Z}^d)$ the random field corresponding to the considered $d+1$-dimensional Ising state, we want to study the d-dimensional random field Y with

$$Y(i)=X((i,0)), \qquad i\in\mathbb{Z}^d \tag{4.4}$$

Obviously, the distribution of Y is the one induced from that of X. Writing ν_β^h and $\nu_\beta^\pm$ (or, ν_β in general) for this induced law from, respectively, the μ_β^h and $\mu_\beta^\pm$ we have for example

$$\int f(\xi)\, d\nu_\beta^+(\xi)\equiv\nu_\beta^+(f)=\mu_\beta^+(f)\equiv\int f(\sigma)\, d\mu_\beta^+(\sigma) \tag{4.5}$$

for the expectation of any function f which is $\mathscr{F}$-measurable (depends only on the $\sigma_{(i,0)}$, $i \in \mathbb{Z}^d$). In particular, for $\beta > \beta_c$ we have $\nu_\beta^+(Y(0)) = \mu_\beta^+(X(0)) \equiv m^*(\beta) > 0$ (the spontaneous magnetization). Similarly, the truncated correlations (or covariances) within the layer Ξ

$$\nu_\beta(f; g) \equiv \nu_\beta(fg) - \nu_\beta(f)\, \nu_\beta(g) = \mu_\beta(f; g) \equiv \mu_\beta(fg) - \mu_\beta(f)\, \mu_\beta(g) \quad (4.6)$$

between any two functions f and g depending on a finite number of coordinates in ξ, decay exponentially fast in the distance between the dependence sets of f and g, whenever this is the case in the considered $d+1$-dimensional Ising state μ_β (which is verified away from the critical point $\beta = \beta_c$).

The problem can therefore not be to evaluate the expectation value of specific observables in our restricted state because this can be done starting from the well-known Ising states. Rather, we are interested in some global characterizations of the restricted states ν_β. More specifically, we wish to understand the ν_β as Gibbs measures for some interaction.

In the study of the convergence properties of this interaction potential it will turn out to be useful to know whether for the original Ising measure μ_β there is good decay of correlations close to the surface when on the surface we impose a typical configuration drawn from ν_β. This is the context of the following proposition. For μ_β the Ising measure on $\{-1, 1\}^{\mathbb{Z}^{d+1}}$, define

$$\mu_\beta^\xi(\cdot) := \mu_\beta(\cdot \mid \mathscr{F})(\xi) \quad (4.7)$$

This is defined for ν_β almost every surface configuration $\xi \in \Omega$. In the sequel we will always work with the right-continuous version of this conditional probabilities (see ref. 10, cf. infra).

One has the following result

Proposition 4.1. Suppose that there are constants $C < \infty$, $m > 0$ such that for all $x \in \mathbb{Z}^{d+1}$, $\mu_\beta(X(0); X(x)) \leqslant C \exp(-2m\,|x|)$. Then there is a set $K_0 \subset \Omega$ with $\nu_\beta(K_0) = 1$ such that for all $\xi \in K_0$ there is a length $\ell = \ell(0, \xi) < \infty$ for which

$$|\mu_\beta^\xi(X(0, 1); X(i, 1))| \leqslant C e^{-m\,|i|} \quad (4.8)$$

whenever $|i| > \ell$.

Proof. The crucial step is to observe that (by definition) $\nu_\beta(\mu_\beta^\xi) = \mu_\beta$. Moreover, by the FKG-inequality (positive correlations), $\nu_\beta[\mu_\beta^\xi(X(0))\, \mu_\beta^\xi(X(x))] \geqslant \nu_\beta[\mu_\beta^\xi(X(0))]\, \nu_\beta[\mu_\beta^\xi(X(x))] = \mu_\eta(X(0))\, \mu_\beta(X(x))$ since $\mu_\beta^\xi(X(x))$ is a bounded measurable function non-decreasing in ξ. The constrained measure μ_β^ξ is itself an FKG-measure so that $\nu_\beta(|\mu_\beta^\xi(X(0,1); X(i,1))|) = \nu_\beta[\mu_\beta^\xi(X(0,1); X(i,1))] \leqslant \mu_\beta(X(0,1); X(i,1)) \leqslant C \exp(-2m\,|i|)$. The conclusion (4.8) now follows from standard Borel–Cantelli arguments. ∎

We feel that the left hand side of (4.8) is monotone non-increasing in ξ when we put more plusses) at least when $h \geqslant 0$ and ξ has positive average magnetization. Proving that $\ell(\xi)$ is decreasing in $\xi \in K_0$ seems to ask however for a rather non-trivial extension of a GHS-type inequality.[13]

4.2. Results on Gibbsian Characterizations

As announced in Section 1, we restrict ourselves to results concerning Gibbsian descriptions of restrictions to a layer of Ising states. We first give a summary of results describing the state of the art before Dobrushin's 1995 talk.[5] We then present the results of the Dobrushin program for these restrictions of the Ising model.

The beginning of the study of Ising restrictions was

Theorem 4.1 (Schonmann[36]). In $d=1$ and for $\beta > \beta_c$ there is no translation invariant uniformly absolutely convergent potential for ν_β^+.

In the following, a further decimation $\nu_\beta^{+,b}$, $b = 3, 4,...$ of this ν_β^+ was considered. This measure is obtained as the restriction of the two-dimensional μ_β^+ to $\{0\} \times b\mathbb{Z}$ or, alternatively, as the restriction of ν_β^+ to the decimated integers $b\mathbb{Z}$.

Theorem 4.2 (Lőrinczi, Vande Velde[30]). For sufficiently large β, for all $b = 3, 4,...$, $\nu_\beta^{+,b}$ is a (bona-fide) Gibbs measure.

Decimation of non-Gibbsian measures can thus be (*bona fide*) Gibbsian measures (and the opposite is also true). We can extend this result to random decimations. In other words, we assign a Bernoulli variable $n_i = 0, 1$ to each site $i \in \mathbb{Z}$. The n_i are independent and identically distributed with density p. We consider the restriction $\nu_\beta^{+,(n_i)}$ of μ_β^+ (or ν_β^+) to the (random) set $\{i \in \mathbb{Z} : n_i = 1\}$ of occupied sites.

Theorem 4.3. There is $p_o > 0$ so that for sufficiently large β, for all $p < p_o$, $\nu_\beta^{+,(n_i)}$ is a (bona-fide) Gibbs measure for almost all (n_i).

One can ask what happens when the temperature is large or when the magnetic field is non-zero. While it is rather easy to show that for sufficiently small $\beta > 0$ or for sufficiently large h, the Ising restrictions ν_β^h are Gibbsian, it is less trivial to show the following

Theorem 4.4 (Lőrinczi[26]). For $d=1$, the Ising restriction ν_β^h is Gibbsian whenever $h \neq 0$.

Theorem 4.1 was given a more intuitive proof in ref. 8. In fact, something more was obtained (so called absence of quasi-locality).

Theorem 4.5 (van Enter, Fernandez, Sokal[8]). Take β sufficiently large and $d=1$. Then there does not exist a continuous local specification Γ such that $\nu_\beta^+ \in \mathcal{G}(\Gamma)$, i.e., ν_β^+ is not a (bona fide) Gibbs measure.

This was extended to any dimension by

Theorem 4.6 (Fernandez, Pfister[10]). Take any dimension d and take $\beta > \beta_c$. Then there does not exist a continuous local specification Γ with $\nu_\beta^+ \in \mathcal{G}(\Gamma)$. I.e., ν_β^+ is not a (bona fide) Gibbs measure.

On the positive side, from ref. 10 it also follows that there exists (for all β, h) an everywhere right-continuous local specification $\Gamma = \Gamma_\beta^h$ so that $\nu_\beta^h \in \mathcal{G}(\Gamma)$. We observed via (3.4) (see also ref. 31) that this implies that the corresponding vacuum potential is always convergent (on Ω)). As we have shown again around (3.4), this implies that ν_β^h (including $h=0^+$) is weakly Gibbsian for the (everywhere) convergent vacuum potential. The question about the absolute convergence of the vacuum potential (on a set of measure one) was also solved:

Theorem 4.7 (Dobrushin, Shlosman[6]). For $d=1$ and β sufficiently large, ν_β^+ is weakly Gibbsian for the absolutely convergent vacuum potential.

For the telescoping potential we have the following

Theorem 4.8 (Maes, Vande Velde[35]). For $d=1$ and β sufficiently large, ν_β^+ is weakly Gibbsian for the absolutely convergent telescoping potential.

The next Section will start with a more detailed presentation (and proof) of this last Theorem.

5. PROOFS

We use here the telescoping potential constructed in Section 3 to prove the results of Section 4. The main thing to show is the exponential decay of this potential for large sets which will follow from the fact that it can be expressed as a correlation function in a two dimensional Ising model on a halfplane with a "typical" surface configuration. The specification Γ used in this section will always be the monotone right-continuous specification consistent with the restriction of the plus phase of the two dimensional Ising model to the line $\{(i,0) : i \in \mathbb{Z}\}$. The telescoping potential introduced in Section 3 is here for $j \neq k$

$$U([j,k],\xi) = -\ln \frac{\gamma_{[j,k]}(\xi^{[j,k]} \mid +)\, \gamma_{[j,k]}(\xi^{]j,k[} \mid +)}{\gamma_{[j,k]}(\xi^{]j,k]} \mid +)\, \gamma_{[j,k]}(\xi^{[j,k[} \mid +)} \tag{5.1}$$

Using that Γ is a specification consistent with the restriction of the plus phase of the two-dimensional Ising model, we have

$$-U([j,k],\xi) = \frac{1}{2}(1-\xi_j)(1-\xi_k) \ln \frac{\mu_\beta^{+,\xi[j,k]}[e^{2\beta X(j,1)} e^{2\beta X(k,1)}]}{\mu_\beta^{+,\xi[j,k]}[e^{2\beta X(j,1)}]\, \mu_\beta^{+,\xi[j,k]}[e^{2\beta X(k,1)}]} + \beta(1-\xi_j)(1-\xi_{j+1})\, \delta_{j,k-1} \tag{5.2}$$

where $\mu_\beta^{+,\xi[j,k]}$ is the constrained measure of (4.7). More specifically, consider the events $S_n(\xi_{[j,k]}) \equiv \{X(x) = +1,\ x \in \Lambda_n^c;\ X(x) = \xi_i,\ x = (i,0),\ i \in [j,k];\ X(x) = +1,\ x = (i,0),\ i \notin [j,k]\}$ where Λ_n is an increasing sequence of squares centered around the origin. For a continuous function f on Ω',

$$\mu_\beta^{+,\xi[j,k]}[f] \equiv \lim_n \mu_\beta^{+}[f \mid S_n(\xi_{[j,k]})] = \lim_n \mu_{n,\beta}^{+,\xi[j,k]}(f) \tag{5.3}$$

This, of course, is a function of the ξ_i, $i \in [j,k]$ only, which act as extra boundary conditions. The limit (5.3) is over the finite volume Ising measures $\mu_{n,\beta}^{+,\xi[j,k]}$ with plus boundary conditions outside the square Λ_n and $\xi^{[j,k]}$ boundary conditions in the middle of the square (on $\Xi \cap \Lambda_n$, cutting the square in two equal parts). For $j = k$, we have $U([j,k],\xi) =$

$$U(j,\xi) = (1-\xi_j) \ln \mu_\beta^{+,\xi j}[e^{2\beta X(j,1)}] + \beta(1-\xi_j) \tag{5.4}$$

The extra term for $j = k-1$ and $j = k$ in (5.2) and (5.4) comes from the interaction inside the layer Ξ ($\sim \mathbb{Z}$) and corresponds to the one-dimensional Ising model. One can check that, $\max_\xi |U_A(\xi)| \leqslant 10\beta$ uniformly in A.

Let $\Omega_U = \bigcap_{i \in \mathbb{Z}} \Omega_U^i$ be defined via

$$\Omega_U^i = \Big\{ \omega \in \{+,-\}^{\mathbb{Z}} : \exists \ell_i^+(\omega), \ell_i^-(\omega) < \infty, \forall k > \ell_i^+(\omega), \forall l > \ell_i^-(\omega), 1/k \sum_{j=0}^{k} \omega(i+j) > 8/9,\ 1/l \sum_{j=-l}^{-1} \omega(i+j) > 8/9 \Big\} \tag{5.5}$$

(This notation suggests of course that Ω_U coincides with the tail field set Ω_U of points of absolute convergence of the potential U introduced in Section 2. We show that this is indeed the case.)

Clearly, $\Omega_U \in \mathcal{T}^\infty$ is a translation invariant set in the tail field. It is easy to see that $\nu^+_\beta(\Omega_U) = 1$ whenever β is sufficiently large. In fact, under this condition, for ν^+_β, the $\ell^\pm_i(\cdot)$, $i \in \mathbb{Z}$ are exponential random variables. This follows from the large deviation properties of μ_β for $\beta > \beta_c$.[36]

In the following proposition we show that the potential U satisfies the bound (3.21) of Section 3. For the sake of completeness we repeat the proof of ref. 35.

Proposition 5.1. The potential defined above (see (5.2), (5.4)) is absolutely convergent for all $\xi \in \Omega_U$ (see (5.5)). In particular, there are constants $C = C_\beta < \infty$, $\lambda = \lambda(\beta) > 0$ so that for all $\xi \in \Omega_U$, $k \in \mathbb{Z}$

$$|U([j, k], \xi)| \leqslant C e^{-\lambda |j-k|} \tag{5.6}$$

whenever $|j-k| > \ell^-_k(\xi)$. The assumption (3.22) holds or, for all $\xi \in \Omega_U$, $i \in \mathbb{Z}$, there is a constant $c_\beta(i, \xi) < \infty$

$$\sum_{k \geqslant i} \sum_{j \leqslant i} |U([j, k], \xi)| \leqslant 16\beta \sum_{k \geqslant i : \ell^-_k(\xi) > k-i} \ell^-(k, \xi) + C \left(\frac{e^\lambda}{e^\lambda - 1}\right)^2 \leqslant c_\beta(i, \xi) \tag{5.7}$$

Proof. Looking at (5.2), we see that the crux of the matter consists in proving that uniformly in the size n of the boxes Λ_n

$$\mu^{+, \xi[0,k]}_{n, \beta}[X(0, 1); X(k, 1)] \leqslant C e^{-\lambda k} \tag{5.8}$$

whenever $n > k > \ell^+(0, \xi)$. This was done in ref. 35.

We repeat the two main steps. They were inspired by the proof of some ergodic properties of the plus phase in ref. 4. The first step reformulates the required estimate in terms of a percolation event. Denote by $E_n(0, k)$ the event that there is a path of consecutive nearest neighbor sites $x = (i, j) \in \Lambda_n$, $j > 0$, connecting $x = (0, 1)$ with $x = (k, 1)$ on which $(X(x), X'(x)) \neq (+1, +1)$. Here X and X' are two independent copies of the random field with law $\mu^{+, \xi[0,k]}_{n, \beta}$. Then

$$|\mu^{+, \xi[0,k]}_{n, \beta}(X(0, 1); X(k, 1))| \leqslant 2\mu^{+, \xi[0,k]}_{n, \beta} \times \mu^{+, \xi[0,k]}_{n, \beta}[E_n(0, k)] \tag{5.9}$$

For the second step, we use that there is a finite constant C so that

$$\mu^+_{n, \beta} \times \mu^+_{n, \beta}[E_n(0, k)] \leqslant C e^{-2\beta k} \tag{5.10}$$

for all sufficiently large β, uniformly in the size n (see ref. 4). The argument is now completed by noticing that

$$\mu_{n,\beta}^{+,\xi[0,k]} \times \mu_{n,\beta}^{+,\xi[0,k]}[E_n(0,k)] \leqslant \exp\left[4\beta \sum_{i\in[0,k]} (1-\xi_i)\right] \mu_{n,\beta}^{+} \times \mu_{n,\beta}^{+}[E_n(0,k)] \tag{5.11}$$

The conclusion is that

$$|\mu_{n,\beta}^{+,\xi[0,k]}(X(0,1); X(k,1))| \leqslant C' e^{4\beta \sum_{i=0}^{k} (1-\xi_i)} e^{-2\beta k} \tag{5.12}$$

If $\xi \in \Omega_U$, only one spin out of eight can be minus in $[0, k]$ for $k > l^+(0, \xi)$ and hence the right hand side of (5.12) is then smaller than $e^{4\beta(1/8)k} e^{-2\beta k}$. ▮

Remark 1. Notice that no use was made of a cluster expansion in the proof above. In fact, a naive application of this cluster expansion is quite impossible as it would yield too much; the attempt in ref. 32 failed for that reason. This is similar to the analysis of Gibbs fields for a random interaction in the Griffiths' regime, see e.g., refs. 2, 7, and 14. We must only concentrate on a specific covariance and percolation techniques seem to be rather powerful in such cases.

Remark 2. An important ingredient in the previous proof is in the step (5.11). Therefore it seems that the proof is necessarily restricted to one dimension. This however is not the case. We can prove the decay of the covariance (as in (5.8)) also in higher dimensions. This we will deal with in a future publication.

So far we dealt with Theorem 4.8. We now prove other theorems of Section 4. For the regular decimation we can only prove Theorem 4.2 for $b > 4$ whereas Lőrinczi *et al.* included also $b = 3, 4$.

(a) The case $h \neq 0$ or $T > T_c$ ($\beta < \beta_c$) (Theorem 4.4). In this case we don't need the steps above. The covariance (the left hand side of (5.12)) is exponentially small uniformly in the boundary condition. That is,

$$|\mu_n^{+,\xi}(X(0,1)\, X(k,1)) - \mu_n^{+,\xi}(X(0,1))\, \mu_n^{+,\xi}(X(k,1))| \leqslant C \exp[-\lambda k] \tag{5.13}$$

This is an immediate application of the result that for $h \neq 0$ or for $\beta \leqslant \beta_c$ the two-dimensional Ising model has a completely analytic interaction.[40]

Therefore the telescoping potential is actually uniformly absolutely convergent and thus in thsis case the projection is a (*bona fide*) Gibbs measure.

(b) Regular decimation (Theorem 4.2). In this case we must follow the above analysis with the only change that (5.2) must be slightly changed. We now have

$$-U([j,k],\xi) = \frac{1}{4}(1-\xi_j)(1-\xi_k) \times \ln \frac{\mu_\beta^{+,b,\xi[j,k]}[e^{2\beta[X(j,1)+X(j,-1)]}e^{2\beta[X(k,1)+X(k,-1)]}]}{\mu_\beta^{+,b,\xi[j,k]}[e^{2\beta[X(j,1)+X(j,-1)]}]\,\mu_\beta^{+,b,\xi[j,k]}[e^{2\beta[X(k,1)+X(k,-1)]}]} \tag{5.14}$$

where $\mu_\beta^{+,b,\xi[j,k]}$ is a new constrained measure (very similar to (4.7)). More specifically, consider the events $S_n^b(\xi_{[j,k]}) \equiv \{X(x)=+1,\ x\in \Lambda_n^c;\ X(x)=\xi_i,\ x=(i,0),\ i\in[j,k]\cap b\mathbb{Z};\ X(x)=+1,\ x=(i,0),\ i\in[j,k]^c\cap b\mathbb{Z}\}$ where Λ_n is an increasing sequence of squares centered around the origin. For a continuous function f on Ω',

$$\mu_\beta^{+,b,\xi[j,k]}[f] \equiv \lim_n \mu_\beta^+[f \mid S_n^b(\xi_{[j,k]})] = \lim_n \mu_{n,\beta}^{+,b,\xi[j,k]}(f) \tag{5.15}$$

The final trick of (5.11) cap be repeated but now, since the ξ's live on a decimated lattice, there are even fewer minusses in the interval $[0,k]$. It suffices that $|b\mathbb{Z}\cap[0,k]| < k/4$ to have an exponential decay of the appropriately modified covariance (5.8) uniformly in the ξ, and thus the decimations are Gibbs for $b>4$.

(c) Random decimations. The analysis is as in the previous case. The final trick of (5.11) must now consider

$$4\beta \sum_{i\in[0,k]} n_i(1-\xi_i) \tag{5.16}$$

Uniformly in ξ this is smaller than $8\beta pk$ for k large enough on a set of Bernoulli variables (n_i) of full measure. It is therefore sufficient to choose the density $p<1/4$. Note that the variables n_i do not even need to be independent, they can be chosen according to an ergodic measure λ with $\lambda(n_i)<1/4$. ▮

6. VARIATIONAL PRINCIPLE

6.1. Existence of Thermodynamic Functions

6.1.1. *Energy Density*

We start from the telescoping potential U defined in (5.2). In this section $\Gamma := \{\gamma_V(\sigma \mid \omega) : V \subset \mathbb{Z}\}$ will as always denote the right-continuous specification such that the restriction of the plus phase of the two dimensional Ising model $\nu_\beta^+ \in \mathcal{G}(\Gamma)$. The set $\Omega_U \subset \Omega$ will always denote the tail field set introduced in (5.5) on which the telescoping potential is absolutely convergent. Given a configuration $\sigma \in \Omega_U$ we define

$$f_U(\sigma) := \sum_{A \ni 0} \frac{1}{|A|} U_A(\sigma) \tag{6.1}$$

Given a probability measure μ on Ω such that $\mu(\Omega_U) = 1$ and $f_U \in L^1(\mu)$, we define

$$e_\mu^U := \int f_U(\sigma)\, d\mu(\sigma) \tag{6.2}$$

Next we introduce the interaction energy in a finite volume $V \subset \mathbb{Z}$:

- free boundary conditions:

$$H_V^f(\sigma) := \sum_{A \subset V} U(A, \sigma) \tag{6.3}$$

- boundary condition ω:

$$H_V^\omega(\sigma) := \sum_{A \cap V \neq \varnothing} U(A, \sigma_V \omega_{V^c}) \tag{6.4}$$

The last sum is well-defined whenever $\omega \in \Omega_U$. For the potential U constructed in Section 3 we show in this subsection that the expectation of the (interaction) energy density exists and equals e_μ^U for a certain class of measures and boundary conditions.

In what follows we use $\mathcal{T}$ to denote the set of translation invariant probability measures on Ω. The symbols C, K, c, λ will always be constants whose values can vary from place to place. We still need the following definitions (cf. Section 5):

1. For $i \in \mathbb{Z}$, $1 \leqslant \alpha < 9/8$, $\sigma \in \Omega$, put

$$l_i^{\alpha,+}(\sigma) := \min \left\{ n \in \mathbb{N} : \forall k \geqslant n \quad \frac{1}{k} \sum_{j=0}^{k-1} \sigma_{i+j} \geqslant \alpha \frac{8}{9} \right\} \tag{6.5}$$

and

$$l_i^{\alpha,-}(\sigma) := \min \left\{ n \in \mathbb{N} : \forall k \geqslant n \quad \frac{1}{k} \sum_{j=0}^{k-1} \sigma_{i-j} \geqslant \alpha \frac{8}{9} \right\} \tag{6.6}$$

One must identify $l_i^{1,+/-}(\sigma)$ with the (abstract) $l_i(\sigma)$ introduced before (cf. Proposition 4.1 and the second remark after Proposition 3.1.)

2. Let $\mathscr{M}_\alpha$ denote the set of probability measures $\mu \in \mathscr{T}$ for which $\mu[l_0^{\alpha,+/-}(\eta) > n] \leqslant e^{-cn}$, $n \in \mathbb{N}$, for some $c > 0$, and

$$\mathscr{M} := \bigcup_{\alpha > 1} \mathscr{M}_\alpha \tag{6.7}$$

Notice that for $\mu \in \mathscr{M}$, e_μ^U of (6.2) is well-defined.

3. Finally we put,

$$\Omega_\alpha := \{\omega \in \Omega \; \exists \varepsilon > 0, \exists N \in \mathbb{N} : \forall i \in \mathbb{Z} \text{ with } |i| \geqslant N, l_i^{\alpha,+/-}(\omega) \leqslant |i|^{1/(3+\varepsilon)}\} \tag{6.8}$$

Note that $\mu \in \mathscr{M}_\alpha \Rightarrow \mu(\Omega_\alpha) = 1$. Also if $\alpha < \beta$ then $\Omega_\alpha \supset \Omega_\beta$ and $\forall \alpha > 1$, $\Omega_\alpha \supset \Omega_U$. The class of measures $\mathscr{M}$ has to be thought of as the analogue of the class of tempered measures in the context of unbounded spin systems (cf. ref. 24, Definition 4.1).

Proposition 6.1 (Energy density for free boundary conditions). Let $\mu \in \mathscr{M}$ and U the potential defined in (3.13). Then,

$$e_\mu^U = \lim_{V \uparrow \mathbb{Z}} \frac{1}{|V|} \mu(H_V^f) \tag{6.9}$$

Proposition 6.2 (Energy density for fixed boundary condition). Let $\mu \in \mathscr{M}_\alpha$ and $\omega \in \Omega_\alpha$ for some $\alpha > 1$. Then,

$$e_\mu^U = \lim_{V \uparrow \mathbb{Z}} \frac{1}{|V|} \mu(H_V^\omega) \tag{6.10}$$

Proof of Proposition 6.1. Consider a sequence of intervals $V_n = \{-n,..., n\}$. Then by translation invariance of μ

$$\lim_{V \uparrow \mathbb{Z}} \left| \left[\frac{1}{|V|} \mu(H_V^f) - e_\mu^U \right] \right|$$

$$= \lim_{n \to \infty} \frac{1}{2n+1} \left| \mu \left[\sum_{A \subset V_n} U(A, \sigma) - \sum_{i \in V_n} \sum_{A \ni i} \frac{1}{|A|} U(A, \sigma) \right] \right|$$

$$\leqslant \lim_{n \to \infty} \frac{1}{2n+1} \left| \mu \left[\sum_{A \cap V_n \neq \varnothing, A \cap V_n^c \neq \varnothing} U(A, \sigma) \right] \right| \tag{6.11}$$

From Section 3 we know that

- $U(A, \sigma) = 0$ if A is not an interval
- we have the upperbounds

$$|U([i, j], \sigma)| \leqslant C.I[l_i^+(\sigma) \geqslant j-i] + C' \mathrm{e}^{-\lambda(j-i)} \tag{6.12}$$

$$|U([i, j], \sigma)| \leqslant C.I[l_j^-(\sigma) \geqslant j-i] + C' \mathrm{e}^{-\lambda(j-i)} \tag{6.13}$$

(cf. Proposition 5.1).

Inserting the exponential term of the RHS of (6.12) in the right hand side of (6.11) gives

$$\lim_{n \to \infty} \frac{C'}{2n+1} \left(\sum_{i=-\infty}^{n} \sum_{j=n+1}^{\infty} + \sum_{i=-\infty}^{-n-1} \sum_{j=-n}^{n} \right) \mathrm{e}^{-\lambda(j-i)} = 0 \tag{6.14}$$

and we are left with two sums, of which we treat only the first one, the second one can be done in an analogous way. We abbreviate in what follows $l_i := l_i^+$ and $l_i^\alpha := l_i^{\alpha, +}$.

$$\lim_{n \to \infty} \frac{C}{2n+1} \mu \left(\sum_{i=-\infty}^{n} \sum_{j=n+1}^{\infty} I[l_i(\sigma) \geqslant j-i] \right)$$

$$= \lim_{n \to \infty} \frac{C}{2n+1} \sum_{i=-\infty}^{n} \mu([l_i(\sigma) - (n-i)].I[l_i(\sigma) \geqslant n+1-i])$$

$$= \lim_{n \to \infty} \frac{C}{2n+1} \sum_{i=-\infty}^{n} \sum_{M=1}^{\infty} M\mu(l_i(\sigma) = M+n-i)$$

$$\leqslant \lim_{n \to \infty} \frac{C}{2n+1} \sum_{i=-\infty}^{n} \sum_{M=1}^{\infty} M \mathrm{e}^{-(M+n-i)c} = 0 \tag{6.15}$$

Proof of Proposition 6.2.

$$\lim_{V\uparrow\mathbb{Z}}\left[\frac{1}{|V|}\mu(H_V^{\omega})-e_\mu^U\right]$$

$$=\lim_{V\uparrow\mathbb{Z}}\frac{1}{|V|}\mu\left(\sum_{A\cap V\neq\varnothing,\,A\cap V^c\neq\varnothing}[U(A,\sigma_V\omega_{V^c})-U(A,\sigma)]\right)\tag{6.16}$$

The second term goes to zero as $V\uparrow\mathbb{Z}$ by the proof of Proposition 6.1. Again we take intervals $V_n=\{-n,..., n\}$ and we use the bound (6.12). The contribution of the exponential part goes to zero so we only have to show that

$$\lim_{n\to\infty}\frac{1}{2n+1}\mu_n\left(\sum_{i=-\infty}^{n}[l_i(\sigma_{V_n}\omega_{V_n^c})-(n-i)].I[l_i(\sigma_{V_n}\omega_{V_n^c})>n-i]\right)=0\tag{6.17}$$

Abbreviate $\sigma\omega:=\sigma_{V_n}\omega_{V_n^c}$ and $f(n):=n^{1/(3+\varepsilon)}$. Fix $\omega\in\Omega_\alpha$, $\mu\in\mathscr{M}_\alpha$ and $n>0$ large enough such that $\bar{l}_n(\omega)\leqslant f(n)$. We distinguish two cases.

Case 1. $l_i^\alpha(\sigma)\leqslant(n-i)$: Here we first show that the set $\{\sigma\in\Omega: l_i(\sigma\omega)>(n-i), l_i^\alpha(\sigma)\leqslant(n-i)\}$ is not empty for only a limited number of i's (for n and α fixed). Indeed, since $l_i^\alpha(\sigma)\leqslant(n-i)$ we have

$$\sum_{k=i}^{n}\sigma(k)\geqslant\alpha\tfrac{8}{9}(n-i+1)\tag{6.18}$$

and thus $l_i(\sigma\omega)>(n-i)$ can only happen if there is a $p>1$ such that

$$\alpha\tfrac{8}{9}(n-i+1)+\sum_{k=n+1}^{n+p}\omega(k)<\tfrac{8}{9}(n-i+1+p)\tag{6.19}$$

Therefore such a p cannot be too small:

$$\alpha\tfrac{8}{9}(n-i+1)-p<\tfrac{8}{9}(n-i+1+p)\tag{6.20}$$

i.e., $p>K(n-i+1)$ where $K=8(\alpha-1)/17>0$. On the other hand since $\omega\in\Omega_\alpha$, $p\geqslant f(n+1)$ would imply

$$\sum_{k=n+1}^{n+p}\omega(k)>\alpha\tfrac{8}{9}p\tag{6.21}$$

therefore (6.19) can only be satisfied if $p < f(n+1)$. Combining the two inequalities we obtained for p we get

$$(n-i+1) < K^{-1} f(n+1) \tag{6.22}$$

and for $l_i^\alpha(\sigma) \leqslant (n-i)$ and $\omega \in \Omega_\alpha$,

$$l_i(\sigma\omega) \leqslant (n-i+1) + f(n+1) \tag{6.23}$$

We can thus estimate

$$\begin{aligned}
&\sum_{i=-\infty}^{n} \mu_n[[l_i(\sigma\omega) - (n-i)]\, I[l_i(\sigma\omega) > (n-i)] \mid l_i^\alpha(\sigma) \leqslant (n-i)] \\
&\leqslant \sum_{i=n-K^{-1}f(n+1)+1}^{n} \mu_n[l_i(\sigma\omega) - (n-i) \mid l_i^\alpha(\sigma) \leqslant (n-i)] \\
&\leqslant \sum_{i=n-K^{-1}f(n+1)+1}^{n} \sum_{j=1}^{f(n)} j = O((f(n))^3)
\end{aligned} \tag{6.24}$$

Case 2. $l_i^\alpha(\sigma) > (n-i+1)$: If $\omega \in \Omega_\alpha$, then, for $p > f(n+1)$,

$$\sum_{j=n+1}^{n+p} \omega(j) > \alpha \tfrac{8}{9} p \tag{6.25}$$

Therefore

$$\sum_{j=i}^{n} \sigma(j) + \sum_{j=n+1}^{n+p} \omega(j) \geqslant -(n-i+1) + \alpha \tfrac{8}{9} p \tag{6.26}$$

Hence

$$-(n-i+1) + \alpha \tfrac{8}{9} \geqslant ((n-i+1) + p)\, \tfrac{8}{9} \tag{6.27}$$

implies $l_i(\sigma\omega) \leqslant p + (n-i+1)$. I.e., if $p \geqslant K^{-1}(n-i+1)$, then $l_i(\sigma\omega) \leqslant p + n - i + 1$, and thus we conclude

$$l_i(\sigma\omega) \leqslant (n-i+1)(1+K^{-1}) \tag{6.28}$$

On the other hand, since $\mu \in \mathscr{M}_\alpha$,

$$\mu(l_i^\alpha(\sigma) > (n-i)) \leqslant e^{-c(n-i)} \tag{6.29}$$

Combining (6.28) and (6.29) we get

$$\sum_{i=-\infty}^{n} \mu_n((l_i(\sigma\omega)-(n-i))\, I[l_i(\sigma\omega)>(n-i)] \mid l_i^{\alpha}(\sigma)>(n-i)) \times \mu_n(l_i^{\alpha}(\sigma)>(n-i)) \leqslant \sum_{i=-\infty}^{n} K(n-i)\, e^{-c(n-i)} = O(1) \tag{6.30}$$

Conditioning respectively on $l_i^{\alpha}(\sigma)\leqslant(n-i)$, $l_i^{\alpha}(\sigma)>(n-i)$, and using (6.24) (6.29), (6.30) we arrive at

$$\sum_{i=-\infty}^{n} \mu_n[(l_i(\sigma\omega)-(n-i))\, I[l_i(\sigma\omega)>(n-i)]] \leqslant O((f(n))^3) \tag{6.31}$$

and hence (6.17) follows. ∎

6.1.2. Pressure

Proposition 6.3. Let U be the potential defined in (3.13). The pressure (or free energy density)

$$P(U) := \lim_{V\uparrow\mathbb{Z}} \frac{1}{|V|} \log Z_V^f \tag{6.32}$$

exists, where

$$Z_V^f \equiv \sum_{\sigma\in\Omega_V} \exp\left[-\sum_{A\subset V} U(A,\sigma)\right] \tag{6.33}$$

is the finite volume (V) partition function with free boundary conditions.

Proof.

$$\sum_{A\subset V} U(A,\sigma) = H_V^f(\sigma) = \log \frac{\gamma_V(+\mid+)}{\gamma_V(\sigma^V\mid+)} \tag{6.34}$$

Therefore

$$Z_V^f = \frac{1}{\gamma_V(+\mid+)} \tag{6.35}$$

The existence of $\lim_V (1/|V|) \log \gamma_V(+ \mid +)$ follows from standard arguments for the two dimensional Ising model. Indeed,

$$\lim_{n\uparrow\infty} \frac{1}{|V_n|} \log \gamma_{V_n}(+ \mid +) = \lim_{n\uparrow\infty} \frac{1}{|V_n|} \log \frac{Z_{\Lambda_n}}{Z_{\Lambda_n \cup \tilde{V}_n}} \tag{6.36}$$

Here Z_Λ denotes the partition function of the two dimensional Ising model for volume Λ and +-boundary conditions outside Λ, $\Lambda_n = \Lambda_{1,n} \cup \Lambda_{2,n}$, where

$$\Lambda_{1,n} := \{(i, j) \in \mathbb{Z}^2 : |(i, j)| \leqslant n, j > 0\} \tag{6.37}$$

$$\Lambda_{2,n} := \{(i, j) \in \mathbb{Z}^2 : |(i, j)| \leqslant n, j < 0\} \tag{6.38}$$

and

$$\tilde{V}_n := \{(i, 0) : i \in V_n\} \tag{6.39}$$

The existence of the limit in (6.36) is thus standard and can be calculated from the cluster expansion.[11] It is equal to minus the free energy density of the two dimensional Ising model plus a surface contribution. ▮

Proposition 6.4. Let $P(U)$ be as in Proposition 6.3 and define

$$Z_V^\omega := \sum_{\sigma_V} \exp\Big[- \sum_{A \cap V \neq \varnothing} U(A, \sigma_V \omega_{V^c}) \Big] \tag{6.40}$$

Then $\forall \alpha > 1 \; \forall \omega \in \Omega_\alpha$

$$P(U) = \lim_{V\uparrow\mathbb{Z}} \frac{1}{|V|} \log Z_V^\omega \tag{6.41}$$

Proof.

$$\frac{1}{|V|} \log \frac{Z_V^f}{Z_V^\omega} = \frac{1}{|V|} \log \mu_V^{\omega, U} \left(\exp\Big[\sum_{A \cap V \neq \varnothing, A \cap V^c \neq \varnothing} U(A, \sigma_V \omega_{V^c}) \Big] \right) \leqslant 0 \tag{6.42}$$

where $\mu_V^{\omega, U}$ is the measure introduced in Section 2, (2.15). The inequality in (6.42) follows from the fact $U(A, \cdot) \leqslant 0 \; \forall A$, $|A| \geqslant 2$. This follows from the expression (5.2) for the potential, and the positivity of correlations for monotonic functions. On the other hand by Jensen's inequality (take $V_n = \{-n,..., n\}$)

$$\liminf_n \frac{1}{2n+1} \log \mu_{V_n}^{\omega, U}\left(\exp\left[\sum_{A \cap V \neq \varnothing, A \cap V^c \neq \varnothing} U(A, \sigma_V \omega_{V^c})\right]\right)$$
$$\geqslant \liminf_n \frac{1}{2n+1} \mu_{V_n}^{\omega, U}\left(\sum_{A \cap V \neq \varnothing, A \cap V^c \neq \varnothing} U(A, \sigma_V \omega_{V^c})\right) = 0 \tag{6.43}$$

as can be obtained from the proof of Proposition 6.2. ∎

6.2. First Part of the Variational Principle

For every finite volume V, every $\omega \in \Omega_U$, every $\mu \in \mathscr{T}$, the following holds

$$S_V(\mu \mid \gamma_V(. \mid \omega)) = -S_V(\mu) + \mu(H_V^\omega) + \log Z_V^\omega \geqslant 0 \tag{6.44}$$

where $S_V(\mu)$ is the entropy of the measure μ, defined as

$$S_V(\mu) := -\sum_{\sigma \in \Omega_V} \mu_V(\sigma) \log \mu_V(\sigma) \tag{6.45}$$

and $S_V(\mu \mid \nu)$ is the relative entropy of the measure μ with respect to the measure ν, defined as

$$S_V(\mu \mid \nu) := \sum_{\sigma \in \Omega_V} \mu_V(\sigma) \log \frac{\mu_V(\sigma)}{\nu_V(\sigma)} \tag{6.46}$$

if μ is absolutely continuous with respect to ν and $S_V(\mu \mid \nu) = +\infty$ otherwise (we make the convention $0 \log 0 = 0$). We still need the following notation:

$$\nu\gamma_V(f) := \int d\nu(\omega) \sum_{\sigma_V} \gamma_V(\sigma_V \mid \omega)\, f(\sigma_V \omega_{V^c}) \tag{6.47}$$

Theorem 6.1. 1. Let $P(U)$ be as in Proposition 6.3. Then

$$P(U) = \sup_{\mu \in \mathscr{M}} [s(\mu) - e_\mu^U] \tag{6.48}$$

where $s(\mu) := \lim_V (1/|V|)\, S_V(\mu)$ is the entropy density.

2. Let $\omega \in \Omega_\alpha$ for some $\alpha > 1$ and $\mu \in \mathscr{M}$. Then
   - the relative entropy density $s(\mu \mid U)$ exists

$$s(\mu \mid U) \equiv \lim_{V \uparrow \mathbb{Z}} \frac{1}{|V|} S_V(\mu_V \mid \gamma_V(\cdot \mid +)) = \lim_{V \uparrow \mathbb{Z}} \frac{1}{|V|} S_V(\mu \mid \gamma_V(\cdot \mid \omega)) \quad (6.49)$$

   - $\forall \nu \in \mathscr{M}$

$$s(\mu \mid U) = \lim_{V \uparrow \mathbb{Z}} \frac{1}{|V|} S_V(\mu_V \mid \nu \circ \gamma_V) \quad (6.50)$$

3. The supremum in (6.48) is reached for $\mu = \nu_\beta^+$.

Proof. Part 1 is a consequence of $S_V(\mu \mid \nu) \geqslant 0$, $\forall \mu, \nu \in \mathscr{T}$ and subsection 6.1. Part 3 is a special case of 2 because $\nu_\beta^+ \in \mathscr{M}$, $\nu_\beta^+ \circ \gamma_{V_n} = \nu_\beta^+$ and $s(\nu_\beta^+ \mid \nu_\beta^+) = 0$. The first statement, (6.49) follows immediately from the previous subsection. To prove (6.50) we need to prove that

- $$\lim_n \frac{1}{2n+1} \int d\nu(\omega)\, \mu(H_n^\omega - H_n^f) = 0 \quad (6.51)$$

- $$\lim_n \frac{1}{2n+1} \int d\nu(\omega) \log \frac{Z_n^f}{Z_n^\omega} = 0 \quad (6.52)$$

To prove that the first limit is zero, we proceed as in the proof of Proposition 6.2. Let c_μ and c_ν be the constants appearing in the definition of $\mathscr{L}$ for μ, resp. ν. We estimate

$$\frac{1}{2n+1} \sum_{j=-\infty}^{n} \mathbb{E}_{\mu \times \nu}([l_j(\sigma, \omega) - (n-j+1)].I[l_j(\sigma, \omega) > n-j+1])$$

$$= \frac{1}{2n+1} \sum_j \sum_{M=1}^{\infty} M\mu \times \nu[l_j(\sigma, \omega) = M + n - j + 1]$$

$$\leqslant \frac{1}{2n+1} \sum_j \sum_{M=1}^{\infty} M\mu \times \nu[l_j(\sigma, \omega) = M + n - j + 1 \mid l_j^\alpha(\sigma) \leqslant n - j + 1]$$

$$+ \frac{1}{2n+1} \sum_j \sum_{M=1}^{\infty} M\mu \times \nu[l_j(\sigma, \omega)$$

$$= M + n - j + 1 \mid l_j^\alpha(\sigma) > n - j + 1].\mathrm{e}^{-(n-j)\, c_\mu}$$

$$
\begin{aligned}
&\leqslant \frac{1}{2n+1} \sum_j \sum_{M=K^{-1}(n-j)}^{\infty} M \mathrm{e}^{-Mc_\nu} + \frac{1}{2n+1} \sum_j \mathrm{e}^{-(n-j)\,c_\mu} \sum_{M=1}^{K(n-j)} M \\
&\quad + \frac{1}{2n+1} \sum_j \mathrm{e}^{-(n-j)\,c_\mu} \sum_{M=K(n-j)+1}^{\infty} M.\nu[l_n^\alpha(\omega) > M] \\
&\leqslant \frac{1}{2n+1} \sum_j \mathrm{e}^{-K^{-1}(n-j)\,c_\nu} + \frac{1}{2n+1} \sum_j O((n-j)^2)\, \mathrm{e}^{-(n-j)\,c_\mu} \\
&\quad + \frac{1}{2n+1} \sum_j \mathrm{e}^{-(n-j)\,c_\mu} \sum_{M=K(n-j)+1}^{\infty} M \mathrm{e}^{-Mc_\nu} \\
&\to 0 \qquad \text{when} \quad n \to +\infty
\end{aligned} \tag{6.53}
$$

This implies (by the argument we used to prove the existence of $P(U)$) that

$$
\lim_{n\to\infty} \frac{1}{2n+1} \int d\nu(\omega) \log \frac{Z_n^f}{Z_n^\omega} = 0 \quad \blacksquare \tag{6.54}
$$

6.3. Second Part of the Variational Principle

The second part of the variational principle characterizes the maximizers of (6.48) as the measures consistent with Γ. Note that any maximizer μ of (6.48) satisfies $s(\mu \mid \nu_\beta^+) = 0$. To conclude $\mu \in \mathscr{G}(\Gamma)$ from $s(\mu \mid \nu_\beta^+)$ we need an extra technical condition:

Theorem 6.2. Suppose that $\mu \in \mathscr{M}$ such that

$$
s(\mu \mid \nu_\beta^+) := \lim_{n\to\infty} \frac{1}{2n+1} S_{\Lambda_n}(\mu \mid \nu_\beta^+) = 0 \tag{6.55}
$$

and

$$
\lim_{n\to\infty} \mu\left(\exp\left[2\beta \sum_{i=-n}^{n} (1-\eta(i))\right]\right) 2^n \exp[-\beta n] = 0 \tag{6.56}
$$

then $\mu \in \mathscr{G}(\Gamma)$.

We will show that ν_β^+ satisfies the hypotheses of Theorem 6.2.

Proof of the Theorem. The first part of the proof follows [12, p. 323] (the variational principle in the regular Gibbs case). We have to show that for all $\Lambda \in \mathscr{L}$ and for every local function g

$$
\mu\gamma_\Lambda(g) = \mu(g) \tag{6.57}
$$

We show that this equality holds for $\Lambda = \{0\}$ and every local function g; (6.57) then follows from translation invariance and the positivity of Γ.

Note that for every $\Delta \in \mathscr{L}$, $S_\Delta(\mu \mid \nu_\beta^+) < \infty$. This implies that for every $\Delta \in \mathscr{L}$ there exists a $\mathscr{F}_\Delta$-measurable function $f_\Delta \geqslant 0$ such that $\mu = f_\Delta . \nu$ on $\mathscr{F}_\Delta$ ($f_\Delta = (d\mu/d\nu_\beta^+)|_{\mathscr{F}_\Delta}$). In ref. 12 it is shown that for every $\varepsilon > 0$ and for every interval $I \ni 0$ there exists a set $\Delta \in \mathscr{L}$ with $I \subset \Delta$ such that

$$\nu_\beta^+(|f_\Delta - f_{\Delta\setminus\{0\}}|) \leqslant \varepsilon \tag{6.58}$$

Fix a local function g and let I be an interval such that $I \ni 0$ and $g \in \mathscr{F}_I$. Given $\varepsilon > 0$ and I, fix Δ as above and define

$$\tilde{g}(\omega) := \sum_{\sigma_0 = +/-} g(\sigma_0 \omega_{0^c})\, \gamma_{\{0\}}(\sigma_0 \mid \omega_{\Delta\setminus\{0\}} +_{\Delta^c}) \tag{6.59}$$

$\tilde{g} \in \mathscr{F}_{\Delta\setminus\{0\}}$. Then

$$\begin{aligned}
&|\mu\gamma_{\{0\}}(g) - \mu(g)| \\
&\quad \leqslant \mu(|\gamma_{\{0\}}(g) - \tilde{g}|) + |\mu(\tilde{g}) - \nu_\beta^+(f_{\Delta\setminus\{0\}}\tilde{g})| \\
&\qquad + \nu_\beta^+(f_{\Delta\setminus\{0\}} \cdot |\tilde{g} - \gamma_{\{0\}} g|) + |\nu_\beta^+(f_{\Delta\setminus\{0\}} \cdot (\gamma_{\{0\}} g - g))| \\
&\qquad + \|g\|_\infty\, \nu_\beta^+(|f_{\Delta\setminus\{0\}} - f_\Delta|) + |\nu_\beta^+(f_\Delta g) - \mu(g)|
\end{aligned} \tag{6.60}$$

Since $\tilde{g} \in \mathscr{F}_{\Delta\setminus\{0\}}$ and $g \in \mathscr{F}_\Delta$, the second and the last term on the right are zero. The fourth term vanishes because $\nu_\beta^+ \in \mathscr{G}(\Gamma)$ and $f_{\Delta\setminus\{0\}} \in \mathscr{F}_{\{0\}^c}$. The fifth term is smaller than $\|g\|_\infty . \varepsilon$ because of the choice of Δ. We are left with the first and the third term. In the quasilocal case they do not cause any trouble because there $\tilde{g} \to \gamma_{\{0\}} g$ in sup-norm as $\Delta \to \mathbb{Z}$, i.e., for Δ large enough, $\|\tilde{g} - \gamma_{\{0\}} g\|_\infty < \varepsilon$. This is not the case here. We put $\Delta := \{-n,..., n\}$ and write

$$\begin{aligned}
&|(\gamma_{\{0\}} g - \tilde{g})(\omega)| \\
&\quad = \left| \sum_{\sigma_0 = +/-} [\gamma_{\{0\}}(\sigma_0 \mid \omega_{0^c}) - \gamma_{\{0\}}(\sigma_0 \mid \omega_{\Delta\setminus 0} +_{\Delta^c})]\, g(\sigma_0 \omega_{0^c}) \right| \\
&\quad \leqslant \|g\|_\infty \sum_{\sigma_0 = +/-} |\gamma_{\{0\}}(\sigma_0 \mid \omega_{0^c}) - \gamma_{\{0\}}(\sigma_0 \mid \omega_{\Delta\setminus 0} +_{\Delta^c})|
\end{aligned} \tag{6.61}$$

We have that

$$\gamma_{\{0\}}(\sigma_0 = + \mid \omega_{0^c}) = \frac{1}{1+\exp\{\sum_{A \ni 0}[U(A, -_0\omega_{0^c}) - U(A, +_0\omega_{0^c})]\}} \tag{6.62}$$

$$\gamma_{\{0\}}(\sigma_0 = + \mid \omega_\Delta +_{\Delta^c}) = \frac{1}{1+\exp\{\sum_{A \ni 0}[U(A, -_0\omega_{\Delta\setminus 0} +_{\Delta^c}) - U(A, +_0\omega_{\Delta\setminus 0} +_{\Delta^c})]\}} \tag{6.63}$$

and an analogous expression for $\sigma_0 = -$.

Use the inequality

$$\left|\frac{1}{1+e^x} - \frac{1}{1+e^y}\right| \leqslant |x-y| \tag{6.64}$$

to obtain for the first term in (6.60)

$$\begin{aligned}
&\mu(|\gamma_{\{0\}}g - \tilde{g}|) \\
&\quad \leqslant 2\,\|g\|_\infty \int d\mu(\omega) \left[\left|\sum_{A \ni 0}[U(A, +_0\omega_{0^c}) - U(A, +_0\omega_{\Delta\setminus 0} +_{\Delta^c})]\right| \right. \\
&\qquad \left. + \left|\sum_{A \ni 0}[U(A, -_0\omega_{0^c}) - U(A, -_0\omega_{\Delta\setminus 0} +_{\Delta^c})]\right|\right] \\
&\quad = 2\,\|g\|_\infty \int d\mu(\omega) \left[\left|\sum_{A \ni 0,\, A \cap \Delta^c \neq \varnothing} U(A, +_0\omega_{0^c})\right| \right. \\
&\qquad \left. + \left|\sum_{A \ni 0,\, A \cap \Delta^c \neq \varnothing} U(A, -_0\omega_{0^c})\right|\right]
\end{aligned} \tag{6.65}$$

where in the second line we used that $U(A, \omega_\Delta +_{\Delta^c}) = 0$ whenever $A \cap \Delta^c \neq \varnothing$. This expression goes to zero when n tends to infinity for every $\mu \in \mathscr{M}$ (cf. the proof of Theorem 6.1).

For the third term in (6.60) it suffices to show that

$$\begin{aligned}
&\limsup_n \int d\nu_\beta^+(\omega) \frac{\mu(\omega_{\Delta\setminus 0})}{\nu_\beta^+(\omega_{\Delta\setminus 0})} \left[\left|\sum_{A \ni 0,\, A \cap \Delta^c \neq \varnothing} U(A, +_0\omega_{0^c})\right| \right. \\
&\qquad \left. + \left|\sum_{A \ni 0,\, A \cap \Delta^c \neq \varnothing} U(A, -_0\omega_{0^c})\right|\right] = 0
\end{aligned} \tag{6.66}$$

Using that $U(A, \cdot) \equiv 0$ if A is not an interval together with the bound (3.23)

$$|U([j,k],\omega)| \leqslant C_1 I[l_j^+(\omega) > k-j] + C_2 I[l_j^+(\omega) \leqslant k-j]\, \mathrm{e}^{-\lambda(k-j)} \tag{6.67}$$

we obtain

$$\begin{aligned}\sum_{A \ni 0,\, A \cap \Delta^c \neq \varnothing} &|U(A, \sigma_0 \omega_{0^c})| \\ &\leqslant \left(\sum_{j=-\infty}^{-n-1} \sum_{k=0}^{+\infty} + \sum_{j=-n}^{0} \sum_{k=n+1}^{+\infty} \right) C_1 I[l_j^+(\sigma_0 \omega_{0^c}) > k-j] \\ &\quad + C_2 I[l_j^+(\sigma_0\omega_{0^c}) \leqslant k-j]\, \mathrm{e}^{-(k-j)\lambda}\end{aligned} \tag{6.68}$$

For the exponential part we are back in the quasilocal case, namely the sum goes to zero as n tends to infinity, uniformly in ω. We continue with $\sigma_0 = +$ and the first sum, i.e., $j < -n$, $k \geqslant 0$. The other sum and $\sigma_0 = -$ can be treated in exactly the same way.

$$\sum_{j=-\infty}^{-n-1} \sum_{k=0}^{+\infty} I[l_j^+(+_0\omega_{0^c}) > k-j] \tag{6.69}$$

$$\begin{aligned} &= \sum_{j=-\infty}^{-n-1} (l_j^+(+_0\omega_{0^c}) - |j|)\, I[l_j^+(+_0\omega_{0^c}) > |j|] \\ &=: \tilde{u}_n(\omega) \end{aligned} \tag{6.70}$$

Note that $\tilde{u}_n(\cdot)$ is monotone non-increasing (cf. (6.5)).

Denote by ρ_β the Bernoulli measure on Ω with $\rho_\beta(\sigma_0 = -) = \mathrm{e}^{-8\beta}$. Then $\nu_\beta^+ \leqslant \rho$ (see ref. 19). Returning to (6.66) we can write, using Cauchy–Schwartz

$$\begin{aligned}\int d\nu_\beta^+(\sigma) &\frac{\mu(\sigma_{\Delta\setminus 0})}{\nu_\beta^+(\sigma_{\Delta\setminus 0})}\, \tilde{u}_n(\sigma) \\ &= \sum_{\sigma_{\Delta\setminus 0}} \nu_\beta^+(\sigma_{\Delta\setminus 0}) \frac{\mu(\sigma_{\Delta\setminus 0})}{\nu_\beta^+(\sigma_{\Delta\setminus 0})} \frac{\rho(\sigma_{\Delta\setminus 0})}{\rho(\sigma_{\Delta\setminus 0})}\, \mathbb{E}_{\nu_\beta^+}(\tilde{u}_n \mid \mathscr{F}_{\Delta\setminus 0})(\sigma_{\Delta\setminus 0}) \\ &\leqslant \left[\sum_{\sigma_{\Delta\setminus 0}} \rho(\sigma_{\Delta\setminus 0}) \left(\frac{\mu(\sigma_{\Delta\setminus 0})}{\rho(\sigma_{\Delta\setminus 0})} \right)^2 \right]^{1/2} \left[\sum_{\sigma_{\Delta\setminus 0}} \rho(\sigma_{\Delta\setminus 0}) (\mathbb{E}_{\nu_\beta^+}(\tilde{u}_n \mid \mathscr{F}_{\Delta\setminus 0})(\sigma_{\Delta\setminus 0}))^2 \right]^{1/2}\end{aligned} \tag{6.72}$$

Now

$$\rho(\sigma_{\Delta\backslash 0}) = \prod_{i=-n,\, i\neq 0}^{n} (1-e^{-8\beta})^{(1+\sigma(i))/2}\,(e^{-8\beta})^{(1-\sigma(i))/2} \tag{6.73}$$

$$\geqslant (1-e^{-8\beta})^{2n}\, e^{-4\beta\sum_{i=-n,\, i\neq 0}^{n}(1-\sigma(i))} \tag{6.74}$$

or

$$\frac{1}{\rho(\sigma_{\Delta\backslash 0})} \leqslant 4^n e^{4\beta\sum_{i=-n,\, i\neq 0}^{n}(1-\sigma(i))} \tag{6.75}$$

for β large. Therefore

$$\left[\sum_{\sigma_{\Delta\backslash 0}} \frac{\mu(\sigma_{\Delta\backslash 0})^2}{\rho(\sigma_{\Delta\backslash 0})}\right]^{1/2} \leqslant \left(\sum_{\sigma_{\Delta\backslash 0}} [2^n e^{2\beta\sum_{i=-n,\, i\neq 0}^{n}(1-\sigma(i))}\,\mu(\sigma_{\Delta\backslash 0})]^2\right)^{1/2}$$

$$\leqslant 2^n \mu(e^{2\beta\sum_{i=-n}^{n}(1-\sigma(i))}) \tag{6.76}$$

On the other hand $\rho \geqslant \nu_\beta^+$ implies that $\forall \Lambda \in \mathscr{L}$

$$\sum_{\sigma_\Lambda} \rho(\sigma_\Lambda)\,\mathbb{E}_{\nu_\beta^+}(f \mid \mathscr{F}_\Lambda)(\sigma_\Lambda) \geqslant \sum_{\sigma_\Lambda} \nu_\beta^+(\sigma_\Lambda)\,\mathbb{E}_{\nu_\beta^+}(f \mid \mathscr{F}_\Lambda)(\sigma_\Lambda) = \mathbb{E}_{\nu_\beta^+}(f) \tag{6.77}$$

for every non-decreasing monotone function f. Or, since $-\tilde{u}_n^2$ is monotone non-decreasing

$$\sum_{\sigma_{\Delta\backslash 0}} \rho(\sigma_{\Delta\backslash 0})[\mathbb{E}_{\nu_\beta^+}(\tilde{u}_n \mid \mathscr{F}_{\Delta\backslash 0})(\sigma_{\Delta\backslash 0})]^2$$

$$\leqslant \sum_{\sigma_{\Delta\backslash 0}} \rho(\sigma_{\Delta\backslash 0})[\mathbb{E}_{\nu_\beta^+}(\tilde{u}_n^2 \mid \mathscr{F}_{\Delta\backslash 0})(\sigma_{\Delta\backslash 0})]$$

$$\leqslant \sum_{\sigma_{\Delta\backslash 0}} \nu_\beta^+(\sigma_{\Delta\backslash 0})[\mathbb{E}_{\nu_\beta^+}(\tilde{u}_n^2 \mid \mathscr{F}_{\Delta\backslash 0})(\sigma_{\Delta\backslash 0})]$$

$$= \int d\nu_\beta^+(\sigma)\,\tilde{u}_n^2(\sigma) \tag{6.78}$$

Putting the pieces together we obtain for the third term in (6.60)

$$\nu_\beta^+(f_{\Delta\backslash 0} \cdot |\tilde{g} - \gamma_\Lambda g|) \leqslant C 2^n \mu(e^{2\beta\sum_{i=-n}^{n}(1-\sigma(i))})[\nu_\beta^+(\tilde{u}_n^2)]^{1/2} \tag{6.79}$$

Lemma 6.1.

$$[\nu_\beta^+(\tilde{u}_n^2)]^{1/2} \leqslant C e^{-\beta n} \tag{6.80}$$

The proof of Lemma 6.1 is straightforward and uses only that $\nu_\beta^+[l_j^+(\omega) > n] \leqslant C e^{-\beta n}$.

Lemma 6.1 together with (6.79) prove the Theorem. ∎

The following Lemma states that for $\mu = \nu_\beta^+$ and β large enough the conditions of the Theorem are fulfilled (as it should be) and a fortiori that the class of μ's for which (6.56) holds is not empty.

Lemma 6.2. For β large,

$$\lim_{n \to \infty} e^{-\beta n} \nu_\beta^+ (e^{2\beta \sum_{i=-n}^n (1-\sigma(i))}) = 0 \tag{6.81}$$

Proof. Note that

$$e^{-\beta n} \nu_\beta^+ (e^{2\beta \sum_{i=-n}^n (1-\sigma(i))}) \leqslant e^{-(\beta/3) n} \nu_\beta^+ (e^{(11/6) \beta \sum_{i=-n}^n (1-\sigma(i))}) \tag{6.82}$$

Obviously, $\nu_\beta^+(f) = \lim_\Lambda \mu_{\beta,\Lambda}^+(f)$ and we can represent $\mu_{\beta,\Lambda}^+(e^{c\beta \sum_{i=-n}^n (1-\sigma(i))})$ for Λ large, in terms of a contour representation. In particular, using the cluster expansion we will prove that for $0 < c < 2$ and $\beta > (\beta_0/(2-c))$ (for some large β_0), $\limsup_n (1/n) \log \nu_\beta^+(e^{c\beta \sum_{i=-n}^n (1-\sigma(i))})$ is uniformly bounded. We adopt the notation of ref. 37 and refer to Sections V.7 and V.8 therein for details. Let $\Gamma_0 = \Gamma_0(\Lambda)$ denote the set of all Ising contours (i.e., contours on the dual lattice) corresponding to configurations in volume Λ. The dependence on Λ will be understood and not explicitly kept as all arguments will turn out to be uniform in the box Λ. Using a contourrepresentation for the partition function, we can write

$$\nu_\beta^+(e^{c\beta \sum (1-\sigma(i))}) \leqslant \lim_\Lambda \frac{\sum_\Gamma \prod_{\gamma \in \Gamma} e^{-2\beta |\gamma|} e^{2c\beta |\text{int}\, \gamma \cap [-n, n]|}}{\sum_\Gamma \prod_{\gamma \in \Gamma} e^{-2\beta |\gamma|}} \tag{6.83}$$

where the sums run over all families of mutually disjoint contours ($\Gamma = \{\gamma_1, ..., \gamma_k : \gamma_i \in \Gamma_0 \text{ and } \gamma_i \cap \gamma_j = \emptyset, \forall i, j = 1, ..., k\}$), $\text{int}\, \gamma$ denotes the set of sites in the interior of γ and we have used that $|\text{int}\, \gamma \cap [-n, ..., n]| \leqslant \frac{1}{2} |\gamma|$.

We will now take the logarithm of the sums appearing in the RHS of (6.83). The cluster expansion enables us to write this logarithm again as a sum but now over connected families of contours where every contour can appear more than once. We therefore change the notation (still following ref. 37) and go over from sets of contours (Γ) to multi-indices A.

We call a map $A: \Gamma_0 \to \mathbb{N}$ a multi-index. $A(\gamma)$ has to be interpreted as the number of times that the contour γ appears. Next we define (for $0 \leqslant r < 2$)

$$z_r(\gamma) := \mathrm{e}^{-(2-r)\,\beta\,|\gamma|} \qquad \text{if} \quad \gamma \cap [-n, n] \neq \varnothing$$

$$:= \mathrm{e}^{-2\beta\,|\gamma|} \qquad \text{otherwise} \tag{6.84}$$

$$z_r^A := \prod_{\gamma \in \Gamma_0} z_r(\gamma)^{A(\gamma)} \tag{6.85}$$

then

$$\log \sum_{\Gamma} \prod_{\gamma \in \Gamma} z_r^{|\gamma|} = \sum_A a^T(A)\, z_r^A \tag{6.86}$$

for suitable coefficients $a^T(A)$ (see [37, p. 466] for their exact expression). Using (6.86) and (6.83) we obtain

$$\log \nu_\beta^+(\mathrm{e}^{c\beta \sum (1-\sigma(i))}) \leqslant \lim_{\Lambda} \sum_A a^T(A)[z_c^A - z_0^A] \tag{6.87}$$

and the multi-indices in the sum must give non-zero weight to at least one contour that intersects $[-n, n]$ otherwise the expression in the square brackets is zero. Using translation invariance, taking the limit $\Lambda \uparrow \mathbb{Z}^2$, dividing by $2n+1$ and taking the limit $n \to \infty$ we get

$$\limsup_{n \to \infty} \frac{1}{2n+1} \log \nu_\beta^+(\mathrm{e}^{c\beta \sum_{i=-n}^{n} (1-\sigma(i))})$$

$$\leqslant 2 \max \left(\left| \sum_{A \ni 0} a^T(A)\, z_c^A \right|, \left| \sum_{A \ni 0} a^T(A)\, z_0^A \right| \right) \tag{6.88}$$

The right hand side of (6.88) is of order $\mathrm{e}^{-(2-c)\,\beta}$ for $c < 2$ and $\beta > (\beta_0/(2-c))$. In other words, for β sufficiently large we have that

$$\nu_\beta^+(\mathrm{e}^{(11/6)\,\beta \sum_{i=-n}^{n} (1-\sigma(i))}) = O(\mathrm{e}^{n\mathrm{e}^{-(1/6)\,\beta}}) \tag{6.89}$$

and hence

$$\lim_{n \to \infty} \mathrm{e}^{-(1/3)\,\beta n} \nu_\beta^+(\mathrm{e}^{(11/6)\,\beta \sum_{i=-n}^{n} (1-\sigma(i))}) = 0 \quad \blacksquare \tag{6.90}$$

6.4. Open Problem

It follows from [36, Section 4.5.2] that

$$s(\nu_\beta^- \mid \nu_\beta^+) > 0 \tag{6.91}$$

provided that this relative entropy exists. This implies that the following two assertions cannot be true together:

- both phases ν_β^+ and ν_β^- are consistent with Γ
-

$$0 = s(\nu_\beta^+ \mid U) = \lim_V \frac{1}{|V|} S_V(\nu_\beta^+ \mid \nu_\beta^- \circ \gamma_V) \tag{6.92}$$

If the first assertion is false, then ν_β^+ and ν_β^- are not almost Gibbsian (i.e., the set of continuity points of Γ has ν_β^+ and ν_β^- measure zero, see ref. 10). It is however believed (though not proved) that ν_β^+ and ν_β^- are actually almost Gibbsian. In that case the second assertion must be false and the limit $\lim_V (1/|V|)\, S_V(\nu_\beta^+ \mid \gamma_V(\cdot \mid \omega))$ is not the same for all boundary conditions ω.

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